2. Team

Problem 2.1. Let $f(z) : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree n. For any $r > 0$, let

$$M(r) := \max_{|z| \le r} |f(z)|. \tag{5}$$

Show that if $R > r > 0$, then

$$\frac{M(R)}{R^n} \le \frac{M(r)}{r^n}. \tag{6}$$

Moreover, "=" holds in (6) if and only if $f(z) = cz^n$ for some constant c.

Problem 2.2. If $b_1 = 1$, $b_2 = 2$ and

$$b_{n+1} = b_n + b_{n-1}$$

for $n \ge 2$. Does the series

$$\sum_{n=1}^{\infty} \frac{1}{b_n}$$

converge? Show all your work.

Problem 2.3. Find all solutions of

$$\begin{cases} \Delta u = 0, & \text{in} \quad B_1 \setminus \{0\} \subset \mathbb{R}^2, \\ u(x) = 0 & \text{on} \quad \partial B_1, \\ u \ge 0, \end{cases}$$

where $B_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$

Problem 2.4. Assume the function $f : [0, 1] \to \mathbb{R}$ is of $C^1[0, 1]$. Prove that the set of critical values of f has measure zero.

Problem 2.5. Let $w(r)$ and $\sigma(r)$ be non-decreasing functions in an interval $(0, R]$. Suppose there holds for all $r < R$

$$w(\tau r) \le \gamma w(r) + \sigma(r) \tag{7}$$

for some $\gamma, \tau \in (0, 1)$. Then for any $\mu \in (0, 1)$ and $r < R$ we have

$$w(r) \le C \left\{ \left(\frac{r}{R} \right)^{\alpha} w(R) + \sigma(r^{\mu} R^{1-\mu}) \right\} \tag{8}$$

where $C = C(\gamma, \tau)$ and $\alpha = \alpha(\gamma, \tau, \mu)$ are positive constants.

Problem 2.6. Let $P_k(x)$ denote the k-th Chebyshev polynomial, i.e. $P_k(\cos\theta) = \cos(k\theta)$ for $k \in \mathbb{N}$. Suppose

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

is a real monic polynomial (i.e. $a_n = 1$) with all roots in $(-1, 1)$. Prove that all the roots of

$$g(x) = \sum_{k=0}^{n} a_k P_k(x) \tag{9}$$

are all real numbers and in $(-1, 1)$.